# Math 206A Lecture 20 Notes

Daniel Raban

November 14, 2018

## 1 Proof of Dehn's Rigidity Theorem

### 1.1 Determinant of the rigidity matrix

Last time, we were proving this theorem.

**Theorem 1.1** (Dehn's rigidity theorem)**.** *Let $P \subseteq \mathbb{R}^3$ be a simplicial convex polytope with graph $\Gamma = (V, E)$. Let $L : E \to R_+$, and let $(\Gamma, L)$ be a framework. Then $(\Gamma, L)$ is statically rigid.*

We had the following lemma. Let $R$ be the rigidity matrix (formally $3n \times 3n - 6$). If the rank of $R$ is $3n - 6$, then $(\Gamma, L)$ is statically rigid.

**Lemma 1.1.** *Let $R'$ be the square submatrix obtained by removing 9 rows and 3 columns (removing 3 vertices and the edges between them). Then $\det(R') \neq 0$.*

*Proof.* Let $a, b, c, d \in V$ be such that $(a, b), (a, c), (a, d) \in E$. Then consider the minor

$$\begin{bmatrix} x_a - x_b & x_a - x_c & x_a - x_d \\ y_a - y_b & y_a - y_c & y_a - y_d \\ z_a - z_b & z_a - z_c & z_a - z_d \end{bmatrix}.$$

The determinant of the matrix will be the product of determinants of the minors.

The proof is in 2 parts.

1. There exists a permutation $\sigma$ such that $\prod R'_{i,\sigma(i)} \neq 0$. This is equivalent to every triangulation having a claw partition (a partition into $K_{1,3}$ bipartite graphs). Proceed by induction. $\Gamma$ is a triangulation, so there exists a vertex of deg $\leq 5$. If $v$ has degree 3, we can find a claw connecting $v$ all its neighbors. If $v$ has degree 4, then pick 3 of the neighbors to get a claw, and then make another claw with the vertex of the remaining neighbor. The $\deg(v) = 5$ case can be split up into various cases we can similarly solve.

2. For all permutations $\sigma$, $\prod R'_{i,\sigma(i)}$ have the same sign. This is equivalent to all claw partitions having the same sign, where the order of the edges in the graph determines the sign (depending on whether it is clockwise or counterclockwise). We claim that every 2 claw partitions are connected by a sequence of triangle moves, where we take a triangle in the graph and reverse the orientation of the triangle's edges in the claw partition. Let $\Pi, \Pi'$ be claw partitions of $\Gamma$, and let $v, v'$ be vertices with an edge between them that differs in $\Pi, \Pi'$. Flip this edge, and keep doing this until you form a cycle. Using Euler's formula, we can show that there is a path through the interior of the cycle. Reverse the two halves of the cycle, one at a time. Then we have reduced to smaller cycles, and we can continue doing this until we get triangles. Then we apply our triangle moves. $\square$